Prime Successors: A Conjecture on the Computational Irreducibility of Primes

Bill Lauritzen

November 26, 2025

Abstract

Prime numbers exhibit well-understood global density patterns yet remain locally irregular and difficult to predict. Although analytic number theory describes how primes are distributed on large scales, it provides no method for determining the next prime after a given one.

This paper proposes a conjecture on the computational irreducibility of prime successors: that any general algorithm for computing p(n+1) from p(n) must, in essence, perform the full computational work required to inspect the intervening integers. The conjecture is framed heuristically, with an emphasis on its conceptual motivation rather than formal model-specific lower bounds. A related conjecture for highly composite numbers is noted as a direction for future research.

1. Introduction

The sequence of primes
2, 3, 5, 7, 11, 13, 17, …
is simple to define but strikingly irregular in its local behavior. While primes become less frequent overall, the exact jump from one prime to the next — the size of the gap p(n+1) - p(n) — shows no known deterministic pattern.

Mathematics provides powerful global results. The Prime Number Theorem and related work describe how many primes lie below a large number, and the Riemann Hypothesis would refine those estimates. Yet none of these results gives a method for computing the next prime after a given prime p(n). In practice, one checks p(n)+1, p(n)+2, p(n)+3, and so on, until the next prime appears. Modern primality tests make this procedure extremely fast, but the basic incremental structure remains.

This conjecture asserts that this incremental search is unavoidable in a computational sense. The claim concerns the local difficulty of finding the next prime, not the global distribution of primes.

2. The Prime Successor Irreducibility Conjecture

The goal of this note is not to develop a fully formal lower-bound theory for prime computation in a specific machine model. Rather, the aim is to state, in simple terms, a conjecture that captures a widely held intuition: that there is no genuine shortcut to the next prime once one knows the current prime.

In this spirit, “inspection” is meant broadly. Here “inspect” includes both direct primality testing and any standard logical method that eliminates composite numbers without explicit testing—for example, skipping even numbers or integers ruled out by modular arithmetic, wheel factorizations, or pre-sieving with small primes. These methods still require logical and computational work on the intervening integers; they do not magically bypass the interval. The conjecture is intended to apply to general algorithms, not to contrived procedures that hard-code a long list of successor primes or otherwise hide the computational cost in unbounded precomputation.

For every n, any algorithm that correctly computes p(n+1) from p(n) must inspect every integer between p(n) and p(n+1), either by testing or by using standard logical eliminations.

The core idea is that no algorithm can bypass this inspection process to shortcut directly to the next prime.

3. Motivation

Primes exhibit a dual nature. Globally, their overall density is described by statistical laws, yet locally they remain unpredictable.

Heuristic models — including Cramer’s model and its refinements — often treat primes as if each integer n is prime with probability 1/log(n). These models capture many large-scale statistical properties but do not give a method for computing the next prime from the current one.

This motivates the view that the step from p(n) to p(n+1) is intrinsically irreducible: the next prime is revealed only by examining the intervening integers.

4. Relation to Computational Irreducibility

Turing’s proof of the Halting Problem showed that some computational processes cannot be shortcut: their outcomes can be determined only by carrying out each step. If the only reliable way to move from p(n) to p(n+1) is to test the intervening integers, then prime succession is a natural example of computational irreducibility appearing within arithmetic itself.

Computational irreducibility also arises in other natural number-theoretic sequences, such as the Möbius and Liouville functions. Framing these examples explicitly in terms of irreducibility may reveal new structural insights.

5. Weaker Variants and Open Questions

Weaker versions of the conjecture may be more approachable. For example:

- The conjecture might hold for "most" primes, in the sense of natural density.
- Occasional exceptions could be allowed, provided they occur only finitely often.
- The conjecture may be studied within particular computational models or resource bounds.

More formal variants of the conjecture could be phrased as lower bounds on the number of operations required to compute p(n+1) from p(n) in standard computational models, but such refinements are left for future work. Even modest progress on any version of these lower-bound questions would be significant, given how little is known unconditionally about the computational hardness of concrete number-theoretic problems.

Open questions include whether formal connections exist between prime-gap statistics, Kolmogorov complexity, or pseudorandomness assumptions used in cryptography.

6. A Parallel Conjecture for Highly Composite Numbers

For every n, any algorithm that correctly computes h(n+1) from h(n) must inspect every integer between them, either by counting its divisors or by using standard logical eliminations.

A justification analogous to the prime case applies: logical eliminations include any provably certain constraints—such as size bounds or divisor-count limits—that rule out candidates without full divisor evaluation. A more detailed analysis of this highly composite variant, including precise formulations in specific computational models, is left for future work.

Acknowledgments

The author thanks Ben Goertzel for helpful comments on an early draft.

The conceptual motivation for this conjecture was influenced by Stephen Wolfram's discussion of computational irreducibility, as developed in *A New Kind of Science* (2002).